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NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

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ON THE REPRESENTATION OF THE STABILITY REGION IN OSCILLATION PROBLEMS WITH THE AID OF THE HURWITZ DETERMINANTS

By E. Sponder

Translation of "Zur Darstellung des Stabilitätsgebietes bei Schwingungsaufgaben mit Hilfe der Hurwitz-Determinanten." Schweizer Archiv, March 1950.

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ON THE REPRESENTATION OF THE STABILITY REGION IN OSCILLATION PROBLEMS WITH THE AID OF THE HURWITZ DETERMINANTS*

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For oscillation phenomena which may also have an unstable course, it is customary to represent the regions where stability or instability prevails in a plane as functions of two parameters x and y.

In order to determine whether stability exists at any point of the plane represented (thus, whether a disturbance of the oscillation phenomenon considered is damped in its course) it must be investigated whether all roots λ of the frequency (characteristic) equation

$$\lambda^n + a_1\lambda^{n-1} + \ldots + a_n = 0$$

have a negative real part at this arbitrary point. The mathematical condition for this is known to be that the n Hurwitz determinants D_1 to D_n which are formed from the coefficients a_1 to a_n and are functions of the two parameters x and y mentioned before are all positive for this point. Resulting from this criterion and completely equivalent to it is the fact that the coefficients a_1 to a_n and only a few certain Hurwitz determinants must be positive.

If, conversely, all roots λ of the frequency equation have a negative real part, all values D_1 to D_n are positive. If one now visualizes the point considered before (for which we assume stability to have been established) as traveling in the representation plane of figure 1, there vary with its parameters x and y also the coefficients a_1 to a_n, the real parts of the roots λ, and finally the n Hurwitz-determinants. If one arrives at a point where for instance a real root

*"Zur Darstellung des Stabilitätsgebietes bei Schwingungsaufgaben mit Hilfe der Hurwitz-Determinanten." Schweizer Archiv, March 1950, pp. 93-96.

disappears, a_n also disappears since $|a_n|$ is the product of the values of all roots; if one reaches, in contrast, a point where the real part of a complex root becomes zero, it can be shown that then D_{n-1} becomes zero. For every case, however, the product $a_n D_{n-1}$ disappears which is nothing else but the Hurwitz determinant of the nth degree

$$a_n D_{n-1} = D_n$$

Thus the important theorem, the proof for which will be presented later, is valid:

The limits of the (usually unique) stable region lie at $D_n = a_n D_{n-1} = 0$.

For graphical representation, it is therefore completely sufficient to plot $D_n = 0$ or more simply $a_n = 0$ and $D_{n-1} = 0$ as separate limiting curves of a region for which stability is known to prevail at an arbitrary point, as illustrated by figure 1. If the limit $D_{n-1} = 0$ is exceeded, the course of the oscillation process is "dynamically" unstable because a damping becomes negative; beyond the limit $a_n = 0$, one usually calls the oscillation process "statically" unstable.

In particular, the following is valid for oscillation phenomena which lead to frequency equations of the 4th degree (reference 1).

The Hurwitz determinant of the 4th degree formed from the coefficients A to E of the frequency equation

$$A\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E = 0$$

reads

$$D_4 = ED_3 = E(BCD - AD^2 - B^2E) = ER$$

with the expression in parentheses known as Routh's discriminant R; the latter is therefore nothing else but the Hurwitz determinant of the

3rd degree. Since the coefficient A is usually +1, it is generally valid as dynamic stability condition that the expression $(BC - D)D - B^2E$ turns out positive. The static stability is then guaranteed in addition by $E > 0$ so that the graphic representation of the region of figure 2 results.

Therewith, every requirement has been met; for it is impossible that within the region denoted as stable a curve $C = 0$ or $D = 0$ could run and perhaps still further reduce this region.

It is therefore completely superfluous to investigate further what sign the other coefficients of the frequency equation have once the boundaries for a region $E = 0$ and $R = 0$ have been fixed; however, and this is important, it must be known that, at an arbitrary point of this region, stability actually prevails. For this it is sufficient to check for a point conveniently situated (for instance on an axis) the signs of the remaining coefficients A to D of the frequency equation which is equivalent with the fact that there all Hurwitz determinants D_1 to D_4 turn out positive. That this is important can, for instance, be recognized from the fact that, in the region indicated on the left in the plane of representation of figure 2, instability may prevail in spite of $E > 0$ and $R > 0$.

The advantage of these recognized facts given above does not result in much simplification for frequency equations of the 4th, 5th, or even 6th degree; however, this advantage may save a great deal of unnecessary calculations. The expressions for the Hurwitz determinants become more and more cumbersome for higher degrees so that it will probably be a very welcome facilitation for the outlining of the stability region if the mere representation of the two curves $a_n = 0$ and $D_{n-1} = 0$ will be sufficient.

There now follows the proof of the theorem given before that the limits of the (usually unique) stable region lie at $D_n = a_nD_{n-1} = 0$.

1. For the present consideration, the existence of at least one stable region is presupposed. Not even one need necessarily always exist; on the other hand, several stability regions, separated from one another, may occur. One can easily make sure of this by considering a frequency equation of the 2nd degree which pertains to an ordinary oscillation

$$\lambda^2 + a_1\lambda + a_2 = 0$$

The coefficients a_1 and a_2 are dependent on two parameters x and y, corresponding to the two coordinate axes of the plane of representation. For simplicity's sake, we shall assume that a_2 everywhere has positive values only; then the sign of a_1 alone is the criterion for the stability, since a_1 is a measure of the damping of the oscillation considered and this oscillation is stable only if a_1 is positive. The limit of stability now depends entirely on the sign of the analytical (or empirical) function of the coefficient a_1 of the two parameters x and y, or, in other words, what form the curves $a_1 = 0$ have in the plane of representation which separate the positive from the negative values. In this manner, one can easily understand that several stability regions may exist but that just as well not even one need exist anywhere in the entire plane of representation.

For frequency equations of higher degree, the probability decreases that simultaneously in several regions all stability conditions found by Hurwitz will be satisfied. Usually, one will deal with only one single stability region which will then be considered more thoroughly.

2. Hurwitz' criterion (reference 2) signifies that a prescribed equation of the nth degree with real coefficients

$$a_0\lambda^n + a_1\lambda^{n-1} + \ldots + a_n = 0 \qquad (a_0 > 0)$$

possesses roots exclusively with negative real parts only when the values of the n-determinants

$$D_k = \begin{vmatrix} a_1 & a_3 & a_5 & \cdot & \cdot & \cdot & a_{2k-1} \\ a_0 & a_2 & a_4 & \cdot & \cdot & \cdot & a_{2k-2} \\ 0 & a_1 & a_3 & \cdot & \cdot & \cdot & a_{2k-3} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_k \end{vmatrix}$$

$$(k = 1,2,\ldots,n)$$

are all positive. Therein, one generally has to put $a_x = 0$ when the subscript x is negative or larger than n.

It shall now be proved that the Hurwitz determinant of the $(n - 1)$th degree D_{n-1} always disappears when the real part of a complex pair of roots is zero.

Following the way Hurwitz pursued in deriving his criterion, one finds the remarkable presupposition that no purely imaginary roots must exist. This, however, is precisely the condition whose influence on a certain Hurwitz-determinant is of special interest. However, for that reason, one need not abandon this presupposition; it is sufficient to interpret D_{n-1} simply as a prescribed calculation rule which one applies, without consideration of its derivation, to the coefficients of an equation of the nth degree.

It will now be expedient to consider (corresponding to the assumption that the real part of a complex pair of roots is to be zero) the equation of the nth degree as split into two factors: one is a quadratic expression with the purely imaginary pair of roots and the other is an expression of the $(n - 2)$th degree, thus

$$\left(\lambda^2 + a\right)\left(a_0\lambda^{n-2} + a_1\lambda^{n-3} + \ . \ . \ . + a_{n-2}\right) = 0$$

therein one may put the coefficient $a_0 = 1$, without impairing the generality; the designation a_0 will, however, be retained. For all remaining coefficients a_1 to a_{n-2}, no additional restriction will be prescribed other than that they are to be real and not all zero (which would be trivial for this statement of the problem). Multiplication of the two factors yields

$$\underbrace{a_0}_{A_0}\lambda^n + \underbrace{a_1}_{A_1}\lambda^{n-1} + \underbrace{\left(a_2 + aa_0\right)}_{A_2}\lambda^{n-2} + \underbrace{\left(a_3 + aa_1\right)}_{A_3}\lambda^{n-3} + \ . \ . \ . +$$

$$\underbrace{aa_{n-2}}_{A_n} = 0$$

The combined coefficients A one now visualizes as substituted in the Hurwitz determinant D_{n-1}, the general term of which has in the sth column and zth line the subscript $x = 2s - z$. Thus, one has there A_{2s-z} or, expressed by the coefficients a, the term

$$a_{2s-z} + aa_{2s-z-2}$$

The value of the determinant D_{n-1} is now extended by multiplying all terms of the first line by a_0 and all terms of the second line by a_1. Furthermore, one makes use of the theorem that a determinant does not change its value if one adds to the elements of one line the elements of another line multiplied by an arbitrary number. This is done by adding to the terms of the first line, which already have been multiplied by a_0, the terms of the third, fifth, . . . line multiplied by a_2, a_4, . . .; thus, one obtains as the general term of the sth column in the first line

$$\sum_{z=1,3,5,\ldots} a_{z-1}\left(a_{2s-z} + aa_{2s-z-2}\right)$$

Likewise, one adds to the terms of the second line already multiplied by a_1 the terms of the fourth, sixth, . . . line multiplied by a_3, a_5, . . .; one then finds as the general term of the sth column in the second line

$$\sum_{z=2,4,6,\ldots} a_{z-1}\left(a_{2s-z} + aa_{2s-z-2}\right)$$

In order to calculate these sums, it is sufficient to note that one must generally put $a_x = 0$ when the subscript x is negative; there then results for a term of the first line

$$\sum_{z=1,3,5,\ldots} a_{z-1}\left(a_{2s-z} + aa_{2s-z-2}\right) = a_0a_{2s-1} + a_2a_{2s-3} + \cdot \cdot \cdot +$$

$$a_{2s-4}a_3 + a_{2s-2}a_1 + a\left(a_0a_{2s-3} + a_2a_{2s-5} + \cdot \cdot \cdot + a_{2s-4}a_1\right)$$

and for a term of the second line

$$\sum_{z=2,4,6,\ldots} a_{z-1}\left(a_{2s-z} + aa_{2s-z-2}\right) = a_1a_{2s-2} + a_3a_{2s-4} + \cdot \cdot \cdot +$$

$$a_{2s-3}a_2 + a_{2s-1}a_0 + a\left(a_1a_{2s-4} + a_3a_{2s-6} + \cdot \cdot \cdot + a_{2s-3}a_0\right)$$

One can see that both sums are of equal magnitude which, however, does not signify anything else but that the corresponding elements of the first two lines are equal and that therefore the Hurwitz-determinant D_{n-1} extended by a_0a_1 identically disappears:

$$a_0a_1D_{n-1} \equiv 0$$

From the derivation follows that one could have extended initially, instead of the first and second line, two arbitrary other odd and even lines by a_{s-1} and then have proceeded further in the same manner; one then would have obtained quite generally

$$a_{even}a_{odd}D_{n-1} \equiv 0$$

Under the obvious assumption that there always exists such a pair of values of the coefficients a_0 to a_{n-2}, the product of which does not disappear, thus

$$D_{n-1} \equiv 0, \text{ Q.E.D.}$$

is valid.

3. From the derivation of this proof, one may further conclude that part of the curve $D_{n-1} = 0$ may run entirely in the unstable region; thus it does not appear there as the required stability limit. Since such a possibility had been pointed out before, the motivation of this noteworthy indication should be mentioned.

In splitting the prescribed equation of the nth degree into two factors, no more detailed data on the coefficients a_0 to a_{n-2} of the expression of the (n - 2)th degree have been given and accordingly none regarding the roots of the equation

$$\left(a_0\lambda^{n-2} + a_1\lambda^{n-1} + \ldots + a_{n-2}\right) = 0$$

either.

These roots may therefore have negative as well as positive real parts in the neighborhood of points of the plane of representation for which D_{n-1} disappears; in other words, they may signify stability and also instability. If there exists at least one root of the former expression of the (n - 2)th degree with a positive real part, this fact signifies that the curve $D_{n-1} = 0$ lies in an unstable region.

Figure 3 is to indicate how, for instance, three roots may vary their position in the complex number plane if their point of reference in the plane of representation shifts beyond the curve $D_{n-1} = 0$. Thus $D_{n-1} = 0$ is not necessarily always the boundary between two regions where stability and instability prevail.

Neither does the sign of the coefficient a in the quadratic expression $(\lambda^2 + a)$ play a role in the derivation of the proof. If a is positive, a purely imaginary pair of roots satisfies the equation $(\lambda^2 + a) = 0$; this case has been considered in particular. However, a could just as well be negative without altering the result of the theorem. This signifies, however, that in the presence of two real opposite-equal roots of the equation of the nth degree, the Hurwitz-determinant of the $(n - 1)$th degree also disappears.

As figure 4 shows, here also an unstable region lies to both sides of the curve $D_{n-1} = 0$ so that again it does not appear as the stability limit.

Thus, these examples show that it is indispensable to make sure whether really all stability conditions on one side of the curve $D_{n-1} = 0$ have been fulfilled; only then that curve forms together with the curve $a_n = 0$ the boundary against the unstable region.

Translated by Mary L. Mahler
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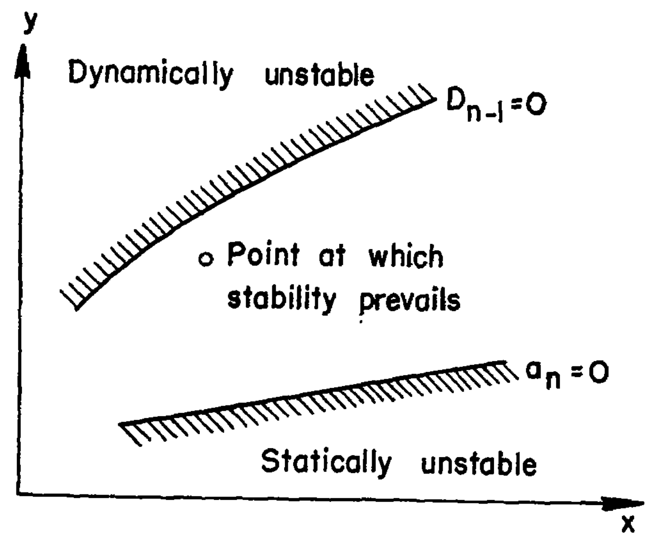

Figure 1.- Travel of the point after establishment of stability.

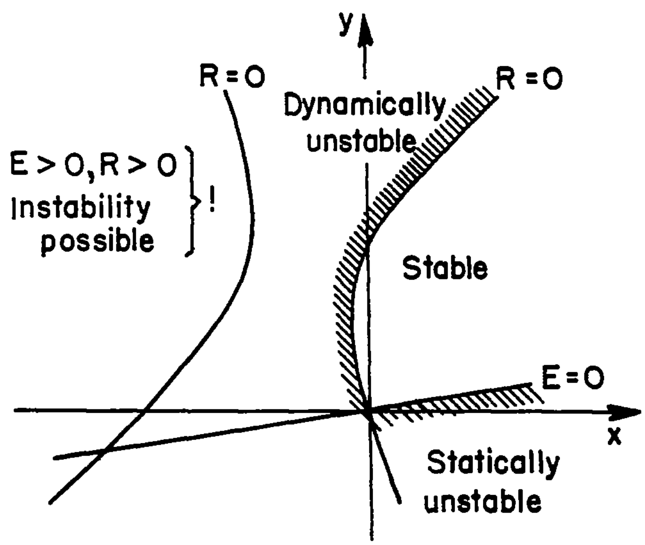

Figure 2.- Graphic representation of the region $E > 0$.

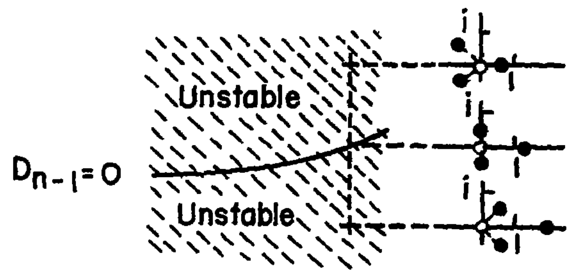

Figure 3.- Change of position of the roots in the complex number plane.

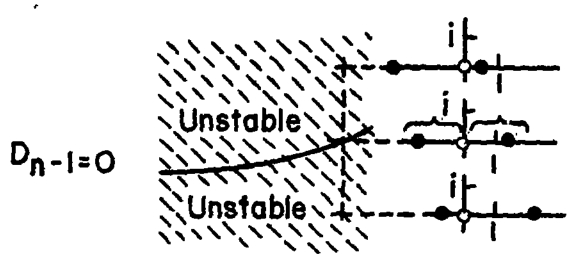

Figure 4.- Unstable region for the curve $D_{n-1} = 0$.

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